Parameterizing the Winner Determination Problem for Combinatorial Auctions

(Extended Abstract)

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ABSTRACT

Combinatorial auctions (CAs) have been studied by the multi-agent systems community for some time, since these auctions are an effective mechanism for resource allocation when agents are self-interested. One challenge, however, is that the winner-determination problem (WDP) for combinatorial auctions is NP-hard in the general case. However, there are ways to leverage meaningful structure in the auction so as to achieve a polynomial-time algorithm for the WDP. In this paper, using the formal scope of parameterized complexity theory, we systematically investigate alternative parameterizations of the bids made by the agents (i.e. the input to the WDP for combinatorial auctions) and are able to determine when a parameterization reduces the complexity of the WDP (fixed-parameter tractable), and when a particular parameterization results in the WDP remaining hard (fixed-parameter intractable). Our results are relevant to auction designers since they provide information as to what types of bidding-restrictions are effective for simplifying the winner determination problem, and which would simply limit the expressiveness of the agents while not providing any additional computational gains.

Categories and Subject Descriptors

F.2 [**Analysis of Algorithms and Problem Complexity**]: Miscellaneous—*combinatorial auctions, winner determination problem*

General Terms

Algorithms and Theory

Keywords

Auction and mechanism design, combinatorial auctions, parameterized complexity

1. PRELIMINARIES

Combinatorial Auctions

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A combinatorial auction consists of a set of n agents, and m items to be auctioned. The set of agents will be denoted by $N = \{1, 2, \ldots, n\}$ and the set of items by M. For any subset $S \subseteq M$, any agent i can place a bid $b_i(S) \in \mathbb{Z}$ on S. We assume that $b_i(S) \geq 0$ for all $S \subseteq M$ and that the agents are self-interested.

An atomic bid, denoted (S, p), includes a set of items $S \subseteq M$ and its bid value, $p \geq 0$. We assume that for each agent i, bids are submitted as a set of atomic bids, $\{(S_{i1}, p_{i1}), \ldots, (S_{ir_i}, p_{ir_i})\}$.

The winner determination problem, which calculates the allocation of goods to agents can be done by solving an integer program, which is NP-hard. We refer the reader to [4] and [1] for a more thorough introduction to CAs and the WDP.

Bid Graphs

Each vertex in a bid graph is a bid, and an edge exists between two vertices if the two bids they represent share an item. We denote the bid for agent i as V_i, which consists of atomic bids $(S_{i1}, p_{i1}), \ldots (S_{ir_i}, p_{ir_i})$ where $S_{ij} \subseteq M$ and $p_{ij} > 0$. For each agent i, we define r_i to be the number of atomic bids in V_i and let M_i represent the total number of items used over all atomic bids, counting each item exactly once. We let R represent the number of distinct subsets of items $S \subseteq M$ that have at least one atomic bid placed by an agent.

By definition, a bid graph is the intersection graph of the distinct atomic bids. The WDP is equivalent to finding a maximum weighted independent set on the constructed graph [4].

Parameterized Complexity

The WDP for the general CA is NP-complete [3]. Even though it is NP-complete, we must research its potential solutions because in practice, some solution is required. There are a number of different approaches to analyzing NP-hard problems, one of which is parameterized complexity theory.

Parameterized complexity theory and the notion of fixed-parameter tractability were developed by Downey and Fellows to further classify intractable problems [2]. By relaxing the requirement that an algorithm runs in polynomial time, the theory allows the running time to be large in terms of one or more *parameters*, provided that it is polynomial with respect to the input size. The goal is to design algorithms that run efficiently if the parameters are sufficiently small, regardless of the size of the input. If such an algorithm exists, the problem it solves is called fixed-parameter tractable

(FPT). If not, we say the parameterization is $W[1]$-hard.

The notions of FPT and $W[1]$ are similar to that of P and NP, and for the purposes of this paper we will not delve further into the many other parameterized complexity classes, but instead refer the reader to Downey and Fellows for a complete description [2].

For further details about the hierarchy and its definition, see Downey and Fellows [2].

2. PARAMETERIZATION

In a CA, we view the numbers of agents, items, and the sizes of the bids as parameters; hence, placing a bound on some or all of these parameters may help restrict the problem space and allow us to derive more efficient solutions. Alternatively, some parameterizations may not reduce the complexity from the NP-hard general problem. Such a parameterization is said to be fixed-parameter intractable. Negative results such as this can be very important in helping us understand what makes the problem difficult to solve. Further, by showing certain parameterizations to be hard, we provide cases to the research community where we know efficient algorithms cannot be found. This allows others to focus on other parameterizations that may yield positive results.

We begin with a simple parameterization of the WDP, which is $W[1]$-complete.

k-WINNER DETERMINATION (k-WD)

Input: A set of agents N, items M, and bids $V = V_1, V_2, \dots V_n$.

Parameter: Positive integer k.

Question: Does there exist a set of mutually disjoint atomic bids $\{(S_1, p_1), (S_2, p_2), \dots, (S_\ell, p_\ell)\}$ such that $\sum_{j=1}^{\ell}(p_j) \geq k$?

Restricting the Graph Class

We consider what happens if we restrict agents to certain types of bids. That is, we restrict each agent to bidding such that the bid graph of the agent's bids maintains a desired structure. The question we then ask is how this structure affects the hardness of the overall combinatorial auction; is the WDP for the auction fixed-parameter tractable, or does it remain $W[1]$-hard?

First we present a general problem definition, in which we restrict our problem using β and parameterize by k, where β is some desired graph class. We require that the graph class of each agent i's bid graph be a graph from class β.

β, k-WINNER DETERMINATION (β, k-WD)

Input: A set of agents N, items M, and bids $V = V_1, V_2, \dots V_n$, where the bid graph generated by V_i belongs to graph class β.

Graph Class: β.

Parameter: Positive integer k.

Question: Does there exist a set of mutually disjoint atomic bids $\{(S_{i_1}, p_{i_1}), (S_{i_2}, p_{i_2}), \dots, (S_{i_\ell}, p_{i_\ell})\}$ such that $\sum_{j=1}^{\ell}(p_{i_j}) \geq k$?

Using different language, Rothkopf *et al.* showed that if β is the class of interval graphs, then β, k-WD is fixed-parameter tractable [3].

PROPOSITION 2.0.1. *If β is the class of chordal graphs, the β, k-WD problem is $W[1]$-hard.*

Given that we cannot restrict β as the class of chordal graphs, a natural question to ask is for which graph classes the β, k-WD problem remains $W[1]$-hard.

A minimal β graph is a graph from graph class β whose size is minimal. It is possible for a minimal β graph to have infinite size, and so our first restriction is that β have a minimal β graph of finite, constant size. We use this restriction in Theorem 2.0.2. The idea is that the minimal β graph has finite, constant size, and thus has a maximum weighted independent set that can be determined in constant time.

THEOREM 2.0.2. *If β has a minimal β graph of finite, constant size and β imposes no restriction on the interaction between the atomic bids of* different *agents, then the resulting β, k-WD problem is $W[1]$-hard.*

3. CONCLUSIONS

For our main result, we restricted graph class β such that the resulting β, k-WD problem remained $W[1]$-hard. This restriction of β is lax enough to allow many different graph classes. It is useful because in the investigation of CAs as they apply to specific economic areas, one may find structure in the bid graphs of individual agents. We show a number of these structures that do not reduce the complexity of the WDP. Knowing which structures are *not* helpful is often just as important as finding ones that lead to fixed-parameter tractability.

Parameterizations are of particular relevance to auction designers since they provide information as to what types of bid-restrictions are effective for simplifying the winner determination problem, and which would simply limit the expressiveness of the agents while not providing any additional computational gains. With each negative result, we find ourselves closer to a more complete characterization of the WDP for CAs and have a deeper understanding of what makes the problem difficult.

The main contribution of this paper is its demonstration of the use of parameterized complexity in the investigation of the WDP for CAs. With parameterized complexity theory, it is possible to parameterize the WDP and discover new, more efficient algorithms, and prove when parameterizations are as hard as any solution for the general WDP. Parameterized complexity could be used to classify many different parameterizations of WDP for CAs to the point where in many "real-world" scenarios where we find a restrictive structure, one could simply look up in a chart to see if such a parameterized version of the CA existed.

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